

Averaging Dynamics on Graphs

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Abstract

We formalize the averaging dynamics on a finite graph, in which every node replaces its value by the average of its neighbours' values: the degree-weighted sum is conserved, the values stay in the initial range, and on a connected graph with an odd closed walk all values converge to the degree-weighted average of the initial values; on a connected bipartite graph they need not converge. The distance to the limit decays like λ^t , with λ the second largest absolute eigenvalue of the random walk (Lovász); in the random sequential version, where one random edge averages its endpoints, the expected state evolves by the matrix $I - L/2m$. On a well-clustered regular graph started from random signs, the sign of the last change recovers the two clusters exactly (Becchetti, Clementi, Natale, Pasquale, Trevisan, SODA 2017).

1 The dynamics

Definition 1 (Averaging). $x_{t+1}(v) = \frac{1}{\deg v} \sum_{u \sim v} x_t(u)$, and $\bar{x} = \sum_v \deg(v) x_0(v) / \sum_v \deg(v)$.

Lemma 2 (Conservation and range). $\sum_v \deg(v) x_t(v)$ does not depend on t ; without isolated nodes, a round never leaves an interval containing all current values.

Proof. Double counting over ordered adjacent pairs; an average of values in an interval stays in it. $\square$

2 Convergence

Lemma 3 (Walk weights). $x_t(v) = \sum_u W_t(v, u) x_0(u)$ with W_t stochastic and $W_t(v, u) > 0$ when a walk of length t joins v and u . On a connected graph with an odd closed walk, some length L joins every pair.

Proof. Induction on t ; for L , fix the parity with the odd closed walk and pad with back-and-forth steps. $\square$

Theorem 4 (Convergence to the degree-weighted average). On a connected graph with an odd closed walk, $x_t(v) \rightarrow \bar{x}$ for every v .

Proof. All entries of W_L are at least some $\delta > 0$, so max – min contracts by $1 - |V|\delta$ every L rounds (Doebelin); the common limit is $\bar{x}$ by conservation. $\square$

Theorem 5 (Bipartite graphs). On a connected bipartite graph with at least two nodes, the values ± 1 of a proper 2-colouring change sign at every round, so they converge nowhere.

Proof. All neighbours of a node have the other colour. $\square$

3 Rate of convergence

Definition 6 (Random walk and its spectrum). $P = D^{-1}A$ is the transition matrix of the random walk, $N = D^{-1/2}AD^{-1/2}$ its symmetric version with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$ (0 if $n < 2$), and $\pi(v) = \deg(v)/2m$ with m the number of edges.

Lemma 7 (Matrix form of the dynamics). $P^t(u, v) = W_t(u, v)$ is the probability that the walk from u is at v after t steps, and $x_t = P^t x_0$. If every degree is positive, P and N have the same characteristic polynomial, so λ is computed from the spectrum of P .

Proof. Induction on t ; $P = SNS^{-1}$ with $S = D^{-1/2}$. $\square$

Theorem 8 (Lovász's bound, Survey Theorem 33). If every degree is positive, $|P^t(u, v) - \pi(v)| \leq \sqrt{\deg(v)/\deg(u)} \lambda^t$ for all nodes u, v and all t .

Proof. $P^t(u, v) = \sqrt{\deg(v)/\deg(u)} N^t(u, v)$ and $\pi(v) = \sqrt{\deg(v)/\deg(u)} s(u)s(v)$ with $s = \sqrt{\pi}$ a unit eigenvector of N for the eigenvalue 1. At most one eigenvalue of N exceeds λ in absolute value, and then it is 1 with eigenvector s ; so N contracts $s^\perp$ by λ , and $N^t(u, v) - s(u)s(v) = \langle Qe_u, N^t Qe_v \rangle$ with $Q = I - ss^\top$ is at most λ^t by Cauchy-Schwarz. $\square$

Lemma 9 (Non-bipartite graphs). On a connected graph with an odd closed walk, $\lambda < 1$.

Proof. An eigenvalue μ of N with $|\mu| = 1$ gives a vector y with $x_t = \mu^t y$ for the dynamics started at y ; by Theorem 4 $\mu = -1$ is impossible and $\mu = 1$ forces y constant, so 1 is a simple eigenvalue and all others have absolute value below 1. $\square$

Corollary 10 (Rate of the averaging dynamics). If every degree is positive, $|x_t(u) - \sum_v \pi(v)x_0(v)| \leq \lambda^t \sum_v \sqrt{\deg(v)/\deg(u)} |x_0(v)|$.

Proof. $x_t(u) - \sum_v \pi(v)x_0(v) = \sum_v (P^t(u, v) - \pi(v))x_0(v)$ and Theorem 8. $\square$

4 Sequential averaging

Definition 11 (Edge step). A step on the edge (i, j) multiplies the state by $W_{ij} = I - \frac{1}{2}(e_i - e_j)(e_i - e_j)^\top$; a run applies the steps of a list of oriented edges in order. For a stochastic matrix P (given as a kernel), $\bar{W}_P = I - \bar{D}/2n + (P + P^\top)/2n$ with $\bar{D}_{ii} = \sum_j (P_{ij} + P_{ji})$, and $\bar{W} = I - L/2m$ with L the Laplacian.

Lemma 12 (Properties of an edge step). $W_{ij}x$ replaces x_i and x_j by their average and keeps the other values; W_{ij} is doubly stochastic and a symmetric projection, $W_{ij}^\top W_{ij} = W_{ij}$.

Proof. Direct computation; $(e_i - e_j)^\top(e_i - e_j) = 2$ for $i \neq j$. $\square$

Theorem 13 (Expected step, Survey equations (4) and (5)). If a uniformly random node i contacts j with probability P_{ij} , then $\mathbb{E}[W_{ij}] = \bar{W}_P$; if a uniformly random oriented edge is selected, then $\mathbb{E}[W_{ij}] = \bar{W}$. On a d -regular graph with $d > 0$ and $P = D^{-1}A$ both equal $(1 - 1/n)I + P/n$.

Proof. Entrywise expansion of $W_{ij}(a, b) = \delta_{ab} - \frac{1}{2}(\delta_{ia} - \delta_{ja})(\delta_{ib} - \delta_{jb})$; there are $\deg(a)$ darts out of a and $2m = nd$ darts in all. $\square$

Theorem 14 (First and second moments). For a uniformly random oriented edge, $\mathbb{E}[W^\top W] = \bar{W}$ and $\mathbb{E}\|Wx\|^2 = x^\top \bar{W}x$; after t independent uniformly random oriented edges, $\mathbb{E}[x_t] = \bar{W}^t x_0$.

Proof. $W^\top W = W$; for the first moment, induction on t and linearity of expectation. $\square$

5 Strong reconstruction

Definition 15 (Clustered regular graph). G is d -regular on two clusters V_1, V_2 of n nodes, each node having b neighbours in the other cluster; $\chi = \mathbb{1}_{V_1} - \mathbb{1}_{V_2}$, $P = A/d$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$, $\lambda = \max_{i \geq 3} |\lambda_i|$. Node v is blue at round t if $x_t(v) \geq x_{t-1}(v)$; a coloring is a strong reconstruction if no color appears in both clusters. $T(n, \delta) = \lceil \log(4n^3) / \log(1 + \delta) \rceil + 1$.

Lemma 16 (Lemma C.1). *If $\lambda < 1 - 2b/d$ and $x \in \{\pm 1\}^{2n}$, then $|x_t(v) - \alpha_1 - \alpha_2(1 - 2b/d)^t \chi(v)| \leq \lambda^t \sqrt{2n}$ with $\alpha_1 = \langle x, \mathbb{1} \rangle / 2n$, $\alpha_2 = \langle x, \chi \rangle / 2n$.*

Proof. $\mathbb{1}$ and χ are eigenvectors with eigenvalues 1 and $1 - 2b/d$, both above λ ; at most two eigenvalues exceed λ in absolute value, so by equality in Bessel's inequality their eigenvectors lie in $\text{span}\{\mathbb{1}, \chi\}$, and P contracts the orthogonal complement of this plane by λ . $\square$

Theorem 17 (Deterministic core). *On a connected clustered graph with $1 - 2b/d > (1 + \delta)\lambda$, if $x \in \{\pm 1\}^{2n}$ and $\langle x, \chi \rangle \neq 0$, then for $t \geq T(n, \delta) = O(\log n / \delta)$, $\text{sgn}(x_{t-1}(u) - x_t(u)) = \text{sgn}(\langle x, \chi \rangle \chi(u))$.*

Proof. $|\langle x, \chi \rangle| \geq 2$ by parity, $b \geq 1$ by connectivity, $d < 2n$, and $(1 + \delta)^{t-1} \geq 4n^3$, so the main term of Lemma 16 dominates the errors. $\square$

Lemma 18 (Lemma B.1, exact form). *For uniform $x \in \{\pm 1\}^{2n}$, $P(\langle x, \chi \rangle = 0) = \binom{2n}{n} / 4^n \leq 1 / \sqrt{\pi n}$.*

Proof. $x \mapsto \{v : x(v) = \chi(v)\}$ maps the zeros bijectively to the n -subsets; Wallis' product. $\square$

Theorem 19 (Strong reconstruction, Theorem 3.2). *With probability at least $1 - 1/\sqrt{\pi n}$, the coloring is a strong reconstruction at every round $t \geq T(n, \delta)$.*

Proof. Theorem 17 and Lemma 18. $\square$