

Reed–Frost Epidemics and Bond Percolation

Leanamics contributors

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Abstract

We formalize the pathwise correspondence between the Reed–Frost (Independent Cascade) epidemic and bond percolation, after Kempe, Kleinberg and Tardos: with one coin per edge, the nodes infected in round t are exactly those at distance t from the initial set in the graph of open edges. Consequently the final outbreak is the set of nodes connected to the initial set by open edges, and with i.i.d. Bernoulli(p) coins the probability of eventual infection is a bond-percolation connection probability. Below the threshold, on graphs of maximum degree d with $p(d-1) \leq 1 - \varepsilon$, every cluster has at most $(10/\varepsilon^2) \log n$ vertices with probability at least $1 - 1/n$ (Becchetti et al.), so a subcritical Reed–Frost epidemic stays small and dies out within $O(\log n)$ rounds. We then formalize the supercritical phase of the Erdős–Rényi random graph $G(n, (1 + \varepsilon)/n)$ (bond percolation on K_n) by the depth-first-search argument of Krivelevich and Sudakov: with probability at least $1 - C/n$ there is a path of length $\varepsilon^2 n/5$ and a component of $\varepsilon n/2$ vertices; read through the coupling, a Reed–Frost epidemic on K_n with $R_0 = pn > 1$ infects $\Omega(n)$ nodes with probability $\Omega(1)$. The development is in the finite layer of the shared `dynamics` package.

We also formalize the deterministic Kermack–McKendrick SIR model: for every solution of $s' = -\beta si$, $i' = \beta si - \gamma i$, $r' = \gamma i$ on $[0, \infty)$ with $r(0) = 0$, the threshold theorem (i initially increases iff $R_0 s(0) > 1$, with $R_0 = \beta/\gamma$), the epidemic peak, and the final-size equation $s_\infty = s(0)e^{-R_0(1-s_\infty)}$, following Hethcote (SIAM Review 2000, Theorem 2.1). This part uses Mathlib’s real analysis.

We also formalize the upper bounds of Doerr and Kozmady’s analysis of general rumor-spreading processes (*Randomized rumor spreading revisited*, ICALP 2017): the exponential growth and shrinking regimes and the total spreading time $\log_{1+\gamma} n + \frac{1}{\rho} \ln n + O(1)$, with exponential tails. Lemma 20 of that paper is false as stated; we prove corrected versions.

We also formalize the duality between the coalescing-branching random walk (COBRA) and the BIPS epidemic of Cooper, Radzik and Rivera (Theorem 4), pathwise and by time reversal of i.i.d. rounds.

1 The model

Definition 1 (Percolated graph and epidemic). Every edge e of a finite graph G carries a coin $\omega(e) \in \{\text{true}, \text{false}\}$; the percolated graph G_ω keeps the edges whose coin is true. In one round, a susceptible node becomes infected if it has an infected neighbour across an open edge, and every infected node recovers. Initially the set I_0 is infected and nobody is recovered. We write $d(v) \in \mathbb{N} \cup \{\infty\}$ for the distance from I_0 to v in G_ω .

2 Distances from a set

Lemma 2 (Distance facts). $d(v) = 0$ iff $v \in I_0$; if u and v are adjacent in G_ω then $d(v) \leq d(u) + 1$; if $d(v) = t + 1$ then v has a neighbour u in G_ω with $d(u) = t$; $d(v) < \infty$ iff v is connected to I_0

in G_ω , and then $d(v) < |V|$.

Proof. The infimum over I_0 is attained; take the second-to-last vertex of a shortest walk; a shortest walk is a path. $\square$

3 Pathwise layers

Theorem 3 (Infected and recovered sets). *For every coin assignment and every t , the nodes infected in round t are those with $d(v) = t$, and the nodes recovered by round t are those with $d(v) < t$.*

Proof. Both statements together, by induction on t , using Lemma 2. $\square$

Corollary 4 (Extinction and final size). *Nobody is infected in round t as soon as every finite $d(v)$ is $< t$, in particular for $t \geq |V|$; the nodes recovered after $|V|$ rounds are those connected to I_0 in G_ω .*

Proof. Immediate from Theorem 3 and $d(v) < |V|$ when finite. $\square$

4 Random coins

Theorem 5 (Reed–Frost $\Leftrightarrow$ bond percolation). *With independent Bernoulli(p) coins, each edge is open with probability p , and the probability that v is eventually infected equals the probability that v is connected to I_0 by open edges.*

Proof. The two events coincide for every coin assignment (Corollary 4). $\square$

The coupling is pathwise because every edge is used at most once: once its coin has been seen from an infected endpoint, that endpoint is recovered for good. This is why a single coin per edge gives the same process, in distribution, as fresh coins in every round (that distributional statement is not formalized here).

5 Subcritical percolation and small outbreaks (EPI-2)

After Becchetti, Clementi, Denni, Pasquale, Trevisan, Ziccardi (arXiv:2103.16398), Theorem 2.3 and its proof, Theorem E.1. Throughout, the degrees of G are at most d , $n = |V|$, and $p(d-1) \leq 1 - \varepsilon$ with $0 < \varepsilon < 1$.

Lemma 6 (Binomial tail and Chernoff bound). *At least t successes among $t(d-1)+1$ independent Bernoulli(p) trials occur with probability at most $\exp(\varepsilon - \varepsilon^2 t/2)$.*

Proof. Markov’s inequality on $e^{\varepsilon X}$ and the bound $\mathbb{E}[e^{\varepsilon X}] \leq \exp(\mu(e^\varepsilon - 1))$ for a sum of independent trials with mean $\mu \leq (1 - \varepsilon)(t + 1)$ (the Chernoff layer of the shared `dynamics` package, before optimizing the tilt: `Distribution.prob_ge_le_exp`), then $(1 - \varepsilon)e^\varepsilon \leq 1 - \varepsilon^2/2$. $\square$

Theorem 7 (Deferred decisions). *The component of s in G_p has more than t vertices with probability at most that of at least t successes in $t(d-1) + 1$ independent Bernoulli(p) trials.*

Proof. For an exploration state (discovered set D , examined pairs X , forced closed), induction on a budget $m \geq \text{frontier}(D, X) + (k-1)(d-1)$: conditioning on the coin of one frontier edge, a closed coin removes that edge from the frontier, and an open one adds a vertex and at most $d-1$ new frontier edges at it while removing the examined one. This is the recursion of the binomial tail. The initial frontier of $\{s\}$ has at most d edges. $\square$

Theorem 8 (Theorems E.1 and 2.3). *The component of any vertex has more than t vertices with probability at most $\exp(\varepsilon - \varepsilon^2 t/2)$, and with probability at least $1 - 1/n$ every component of G_p has at most $(10/\varepsilon^2) \log n$ vertices.*

Proof. Theorem 7 and Lemma 6, then a union bound over the vertices. $\square$

Corollary 9 (Subcritical Reed–Frost and Erdős–Rényi). *With probability at least $1 - 1/n$, the Reed–Frost epidemic with $R_0 = p(d-1) \leq 1 - \varepsilon$ infects at most $|I_0|(10/\varepsilon^2) \log n$ nodes and nobody is infected in round $\lfloor (10/\varepsilon^2) \log n \rfloor$; and every component of $G(n, c/n)$ with $c \leq 1 - \varepsilon$ has at most $(10/\varepsilon^2) \log n$ vertices.*

Proof. The final outbreak is the union of the components of I_0 , and distances inside a component are smaller than its size (Corollary 4). $G(n, c/n)$ is bond percolation on K_n , with $d = n - 1$ and $p(d - 1) \leq c$. $\square$

6 The supercritical giant component

Source: M. Krivelevich and B. Sudakov, *The phase transition in random graphs: a simple proof*, Random Structures & Algorithms 43 (2013), arXiv:1201.6529. Throughout, $G(n, p)$ is the percolated complete graph $(K_V)_\omega$ with i.i.d. Bernoulli(p) coins, $n = |V|$, and $p = (1 + \varepsilon)/n$ is written $pn = 1 + \varepsilon$.

Lemma 10 (Principle of deferred decisions). *If an adaptive strategy queries the coins one at a time, the next coordinate being a function of the answers so far, and never queries a coordinate twice among its first m queries, then its first m answers are i.i.d. Bernoulli(p).*

Proof. The answers equal a list L iff the coordinates queried along L , which are determined by L and distinct, take the values of L : a cylinder event of probability $\prod_k w(L_k)$. $\square$

Definition 11 (Depth-first search fed with answers). The search of Krivelevich–Sudakov (Section 2) with the sets S (done), U (stack) and T (unvisited), stopped before each query; after $U = T = \emptyset$ the remaining pairs are queried.

Lemma 12 (Properties of the search). *Every pair is queried at most once; all pairs between S and T have been queried negatively; U spans a path; $|U| \leq 1 + \sum X_i$ and, while $T \neq \emptyset$, $|S \cup U| \geq \sum X_i$; the vertices of one epoch are connected; when an epoch starts, the explored set D has all its pairs with $V \setminus D$ queried.*

Proof. An invariant preserved by every move and answer; moves without query terminate because $2|S| + |U| \leq 2n$ increases. $\square$

Lemma 13 (Deterministic analysis). *The deterministic steps of the proofs of Theorems 1 and 2 of the paper: $|S \cup U| < n/3$ at time N_0 , $|U| \geq \ell n + 1$, and the current epoch started before t_1 .*

Proof. In each case $|S||T|$ (or $|D|(n - |D|)$) would exceed the number of queries. $\square$

Lemma 14 (Krivelevich–Sudakov, Lemma 1(2), with Chernoff). *Among N_0 i.i.d. Bernoulli(p) trials, the number of successes deviates from $N_0 p$ by $\delta N_0 p$ with probability at most $2e^{-\delta^2 N_0 p/3}$ (Chernoff bounds of FND-3 instead of Chebyshev, as in the paper’s Discussion, item 1); the same holds at every time of a window.*

Proof. Chernoff bounds and the union bound. $\square$

Theorem 15 (Long path; Theorem 1(2)). *For every small enough $\varepsilon > 0$ there is C such that $G(n, (1 + \varepsilon)/n)$ contains a path with at least $\varepsilon^2 n/5$ edges with probability at least $1 - C/n$.*

Proof. Run the search for $N_0 = \lfloor \varepsilon n^2/2 \rfloor$ queries; on the typical event the stack is a path of the required length. $\square$

Theorem 16 (Giant component; Theorem 2 and every ε). *For every small enough $\varepsilon > 0$ there is C such that $G(n, (1 + \varepsilon)/n)$ has a component with at least $\varepsilon n/2$ vertices with probability at least $1 - C/n$; for every $\varepsilon > 0$ there are $c > 0$ and C such that it has a component with at least cn vertices with probability at least $1 - C/n$.*

Proof. No epoch starts between the times $t_1 = \lfloor \eta n^2 \rfloor$ and $N_0 = \lfloor \theta n^2 \rfloor$, so the positive answers in between lie in one component; $\theta = \varepsilon/2$ for the first statement, θ small in terms of ε for the second. $\square$

Corollary 17 (Supercritical Reed–Frost epidemics). *On K_n with $pn = R_0 > 1$, the Reed–Frost epidemic started from any single node infects at least cn nodes with probability at least $q > 0$ for $n \geq n_0$; for $R_0 = 1 + \varepsilon$ with ε small, at least $\varepsilon n/2$ nodes with probability at least $\varepsilon/2 - C/n$.*

Proof. The final outbreak is the component of the initial node (Corollary 4); by symmetry of K_n this node lies in a component of size $\geq k$ with probability at least k/n times the probability that such a component exists. $\square$

7 The Kermack–McKendrick SIR model

Definition 18 (Solutions). Let $\beta, \gamma > 0$ and $R_0 = \beta/\gamma$. A solution is a triple (s, i, r) of real functions such that $t \mapsto (s, i, r)(t)$ is an integral curve on $[0, \infty)$ (one-sided derivative at 0) of the field $(s, i, r) \mapsto (-\beta si, \beta si - \gamma i, \gamma i)$, with $s(0), i(0) > 0$, $r(0) = 0$ and $s(0) + i(0) + r(0) = 1$. Solutions are assumed, not constructed.

Theorem 19 (Invariants). *On $[0, \infty)$: $s + i + r = 1$; $s, i > 0$ and $r \geq 0$; s is strictly decreasing and r strictly increasing; and $s(t) = s(0)e^{-R_0 r(t)}$.*

Proof. The derivatives of $s + i + r$ and of $se^{R_0 r}$ vanish ($R_0 \gamma = \beta$), and a function with zero right derivative is constant. For $i > 0$, compare i with the barrier $b(t) = \frac{i(0)}{2}e^{-\gamma t}$: where they touch, $b' = -\gamma i < \beta si - \gamma i = i'$ since $s > 0$, so $i \geq b$. Monotonicity follows from the signs of $s' = -\beta si$ and $r' = \gamma i$. $\square$

Theorem 20 (Threshold). *If $R_0 s(0) \leq 1$, then i is strictly decreasing on $[0, \infty)$. The function i is strictly increasing on some $[0, \varepsilon]$, $\varepsilon > 0$, iff $R_0 s(0) > 1$.*

Proof. $i' = i(\beta s - \gamma)$ and $s(t) < s(0)$ for $t > 0$. If $R_0 s(0) > 1$, then $\beta s - \gamma > 0$ near 0 by continuity. $\square$

Theorem 21 (Limits and final size). *$i(t) \rightarrow 0$, $s(t) \rightarrow s_\infty$ and $r(t) \rightarrow 1 - s_\infty$; moreover $s_\infty = s(0)e^{-R_0(1-s_\infty)}$, $0 < s_\infty < 1/R_0$, and s_∞ is the unique root of this equation in $(0, 1/R_0]$ and in $(0, 1]$.*

Proof. s and r are monotone and bounded, so $i = 1 - s - r$ converges; if $i \rightarrow L > 0$, the mean value theorem makes r grow by more than 1. Pass to the limit in $s = s(0)e^{-R_0 r}$. If $R_0 s_\infty \geq 1$ then $i' > 0$ forever, contradicting $i \rightarrow 0$. Uniqueness: $\log x - R_0 x$ is strictly increasing on $(0, 1/R_0]$ and strictly decreasing on $[1/R_0, \infty)$, and at $x = 1$ it exceeds its value $\log s(0) - R_0$ at the roots because $s(0) < 1$. $\square$

Theorem 22 (Peak). *If $R_0 s(0) > 1$, there is $t_{\max} > 0$ with $s(t_{\max}) = 1/R_0$ such that i is strictly increasing on $[0, t_{\max}]$ and strictly decreasing on $[t_{\max}, \infty)$, and $i(t_{\max}) = i(0) + s(0) - 1/R_0 - \log(R_0 s(0))/R_0$.*

Proof. Intermediate value theorem between $s(0) > 1/R_0$ and $s_\infty < 1/R_0$; the sign of $\beta s - \gamma$ changes at $t_{\max}$. The value follows from $\log s = \log s(0) - R_0 r$ and $r = 1 - s - i$. $\square$

8 Kurtz's law of large numbers for SIR, in discrete time

Definition 23 (The uniformized SIR chain). Let $\beta, \gamma \in \mathbb{N}$. A configuration assigns to each of N agents a compartment S, I or R ; $X = (S, I, R)/N$ are the scaled counts. One step draws (u, v) uniformly in $[N]^2$ and one of $\beta + \gamma$ equally likely clocks: on one of the β infection clocks an infected u infects a susceptible v , on one of the γ recovery clocks an infected u recovers. This is the continuous-time chain ($S + I \rightarrow 2I$ at rate $\beta SI/N$, $I \rightarrow R$ at rate γI) uniformized at rate $(\beta + \gamma)N$. The deviation probability is the probability that $\|X_k - x(k/((\beta + \gamma)N))\|_\infty > \theta$ for some $k \leq n$.

Lemma 24 (Drift and bounded increments). $\mathbb{E}[X_{k+1} - X_k \mid X_k = x] = F(x)/((\beta + \gamma)N)$ with F the Kermack–McKendrick field, and $\|X_{k+1} - X_k\|_\infty \leq 1/N$.

Proof. Among the $N^2(\beta + \gamma)$ rounds, βIS move an agent from S to I and γNI move one from I to R ; a round changes the compartment of at most one agent. $\square$

Lemma 25 (Maximal Azuma–Hoeffding). *If increments $D(X_j, \rho_j)$ along i.i.d. uniform rounds have zero conditional mean and $|D| \leq c$, their partial sums M_k satisfy $\mathbb{P}(\exists k \leq n, M_k \geq \lambda) \leq e^{-\lambda^2/(2nc^2)}$.*

Proof. Hoeffding's lemma ($\mathbb{E}e^{\theta D} \leq \cosh(\theta c) \leq e^{\theta^2 c^2/2}$ by convexity) makes $e^{\theta M_k - k\theta^2 c^2/2}$ a non-negative supermartingale; Ville's maximal inequality, proved by induction on n peeling off the first round, bounds the probability that it reaches μ by $1/\mu$; take $\theta = \lambda/(nc^2)$. $\square$

Theorem 26 (Law of large numbers). *Let $\beta, \gamma, T > 0$. There are $C, c > 0$ and L such that for every N , every initial configuration, every solution x of the Kermack–McKendrick system started in the simplex and every $\varepsilon > 0$, with probability at least $1 - Ce^{-c\varepsilon^2 N}$, $\|X_k - x(k/((\beta + \gamma)N))\|_\infty \leq L\|X_0 - x(0)\|_\infty + \varepsilon$ for all $k \leq T(\beta + \gamma)N$. Hence, if $X_0 \rightarrow x(0)$, the scaled chain converges to x in probability, uniformly on $[0, T]$.*

Proof. The simplex is invariant under the ODE (first integral for s , integrating factor for i); there F is $(2\beta + \gamma)$ -Lipschitz and bounded by $\beta + \gamma$, so one Euler step errs by $O(h^2)$, $h = 1/((\beta + \gamma)N)$. By Lemma 24 each coordinate of $X_k - X_0 - h \sum_{j < k} F(X_j)$ is a martingale with increments at most $2/N$; on the event where the six one-sided maximal deviations stay below $\delta = \varepsilon/(2L)$, the discrete Grönwall inequality gives the bound with $L = e^{(2\beta + \gamma)T}$, once $\varepsilon N \geq 2LT(2\beta + \gamma)$; Lemma 25 bounds the complement by $6e^{-c\varepsilon^2 N}$. Small N and $\varepsilon \geq 2$ are absorbed in C . $\square$

9 Randomized rumor spreading revisited

We follow Doerr and Kosterlygin, *Randomized rumor spreading revisited* (ICALP 2017; long version arXiv:2303.11150), with the numbering of the long version.

Definition 27 (Rumor-spreading processes). A rumor-spreading process on n nodes is a Markov kernel on the set S of informed nodes under which informed nodes stay informed. For a round started from S , $\text{informProb}(S, x)$ is the probability that x is informed after the round and $\text{cov}(S, x, y)$ the covariance of the events that x and y are. The tail of the spreading time is $\text{notYet}(m, t, S) = \mathbb{P}[\text{fewer than } m \text{ nodes informed after } t \text{ rounds from } S] = \mathbb{P}[T(|S|, m) > t]$; expected times are bounded through all partial sums of $\sum_t \mathbb{P}[T > t]$. Homogeneity (Definition 6) is defined but not assumed.

Definition 28 (Upper growth and shrinking conditions). Upper exponential growth (Definition 9): for $1 \leq |S| = k < fn$, every uninformed node is informed with probability at least $\gamma \frac{k}{n} (1 - a \frac{k}{n} - \frac{b}{\ln n})$ and covariances are at most ck/n^2 . Upper exponential shrinking (Definition 11): when $u = n - |S| \leq gn$, every uninformed node stays uninformed with probability at most $e^{-\rho} + au/n$ and covariances are at most c/u .

Lemma 29 (Lemma 9). *If distinct uninformed nodes have covariance at most c , the number of informed nodes after one round has variance at most its expected increase plus $c(n - |S|)^2$.*

Proof. Expand the variance of a sum of indicators. $\square$

Lemma 30 (Lemma 19). *If every uninformed node is informed with probability at least p in every round started with $\ell \leq |S| < m$ informed nodes, then $\mathbb{P}[T(\ell, m) > r] \leq \frac{n-\ell}{n-m} (1-p)^r$ and $E[T(\ell, m)] \leq \frac{n-\ell}{n-m} \cdot \frac{1}{p}$.*

Proof. The potential $n - |S|$ (set to 0 once $|S| \geq m$) contracts by $1 - p$ in expectation. $\square$

Theorem 31 (Theorem 21: exponential growth). *Under the upper exponential growth conditions, with γ between two positive constants and $af < 1$, from any nonempty S : $\mathbb{P}[T(|S|, fn) > \lceil \log_{1+\gamma} n \rceil + r] \leq Ae^{-\alpha r}$ and $E[T(|S|, fn)] \leq \log_{1+\gamma} n + B$, with constants depending only on the parameters.*

Proof. Phases $k_{j+1} = k_j + E_0(k_j)$ with a round target $E_0(k) = E(k) - Ak^{3/4}$; Cantelli's inequality (Lemma 29) bounds the probability of missing a target, and a phase potential on the iterated kernel replaces the paper's domination by geometric variables. Lemma 30 crosses the last stretch up to fn . $\square$

Lemma 32 (Lemma 20, corrected). *If every round started with $1 \leq |S| < fn$ has $p_k \leq p$ and covariances at most c/n , then (one round) for every $f' \in]f + p(1-f), 1[$, a round from $|S| < fn$ ends with at least $f'n$ informed nodes with probability at most C/n ; and (path form) there is $f' \in]f, 1[$ such that the probability of jumping over $[fn, f'n[$ within t rounds is at most $\frac{C}{n}$ times the expected number of these rounds started below fn , i.e. $O(E[T(|S|, fn)]/n)$.*

Proof. Chebyshev's inequality with Lemma 29 for one round, then a union bound over the rounds along the trajectory. $\square$

The paper's Lemma 20 claims probability $1 - O(1/n)$ of landing in $[fn, f'n]$; this is false: the process that from every $|S| < fn$ informs all nodes with probability c/n (and otherwise none) satisfies the hypotheses and jumps from 1 to n with probability one. See `FORMALIZATION_DIFFERENCES.md`.

Theorem 33 (Theorem 31: exponential shrinking). *Under the upper exponential shrinking conditions, with ρ between two positive constants and $e^{-\rho_{10}} + ag < 1$, from any S with $n - |S| \leq gn$: $\mathbb{P}[T(|S|, n) > \lceil \ln n / \rho \rceil + r] \leq Ae^{-\alpha r}$ and $E[T(|S|, n)] \leq \ln n / \rho + B$.*

Proof. Lemma 30 first brings the number u of uninformed nodes below $g_0 n$; then the potential $\Phi = u + \frac{\beta}{n} u^2$ satisfies $E[\Phi'] \leq e^{-\rho}(1 + K/n)\Phi$ (Lemma 29 for the second moment), which replaces the paper's phase calculus. $\square$

Theorem 34 (Total spreading time). *Under the growth conditions on $[1, fn]$, the shrinking conditions below gn uninformed nodes and a probability at least $p > 0$ of being informed in between, from any nonempty S : $\mathbb{P}[T > \lceil \log_{1+\gamma} n \rceil + \lceil \ln n / \rho \rceil + r] \leq Ae^{-\alpha r}$ and $E[T] \leq \log_{1+\gamma} n + \ln n / \rho + B$.*

Proof. Compose the three exponential tails (Theorem 31, Lemma 30, Theorem 33) by the Markov property at fixed times, and sum the tail. $\square$

10 COBRA and BIPS: the model

We follow Cooper, Radzik and Rivera, *The coalescing-branching random walk on expanders and the dual epidemic process*, PODC 2016 (arXiv:1602.05768).

Definition 35 (Rounds of neighbour choices). In a round r , every vertex x of a finite graph G samples k neighbours $r(x, 0), \dots, r(x, k-1)$. A uniform round is exactly the paper's sampling: k uniform neighbours with replacement, independently for every vertex; t i.i.d. uniform rounds are averaged with `Dynamics.expList`. If G is connected and has at least two vertices, rounds exist.

Proof. Every vertex of a connected graph with at least two vertices has a neighbour. $\square$

Definition 36 (COBRA and BIPS). COBRA from $C_0 = C$: $C_{s+1} = \{r_{s+1}(x, i) : x \in C_s, i < k\}$. BIPS with persistent source v from A_0 : $A_{s+1} = \{v\} \cup \{u : \exists i, r_{s+1}(u, i) \in A_s\}$.

11 COBRA–BIPS duality

Lemma 37 (Time reversal of i.i.d. rounds). *For every functional F of T i.i.d. uniform rounds, $\mathbb{E}[F(r_T, \dots, r_1)] = \mathbb{E}[F(r_1, \dots, r_T)]$.*

Proof. The uniform average over $\text{Fin } T \rightarrow \alpha$ is invariant under $\omega \mapsto \omega \circ \text{rev}$ (the time-reversal lemma of the shared `dynamics` package). $\square$

Lemma 38 (Pathwise duality). *For all rounds $r_1, \dots, r_t$: COBRA from C along $r_1, \dots, r_t$ visits v at some time $s \leq t$ (time 0 included) iff BIPS from $\{v\}$ along $r_t, \dots, r_1$ infects a vertex of C .*

Proof. Induction on t , generalizing C . Both sides say that a chain $x_0 \in C$, $x_j = r_j(x_{j-1}, i_j)$, reaches v within t steps: COBRA unfolds the first round, BIPS the last round of the reversed sequence. $\square$

Theorem 39 (Theorem 4 of Cooper–Radzik–Rivera). *For every finite graph, every k , every vertex v , set C and time t : $\hat{\mathbb{P}}(\text{Hit}_C(v) > t \mid C_0 = C) = \mathbb{P}(C \cap A_t = \emptyset \mid A_0 = \{v\})$. In particular (equation (2)), $\hat{\mathbb{P}}(\text{Hit}_u(v) > t) = \mathbb{P}(u \notin A_t \mid A_0 = \{v\})$. The paper assumes G connected and regular and $k \geq 1$; none of this is needed.*

Proof. Lemma 38 identifies the two events after reversing the rounds, and Lemma 37 says that reversing the rounds preserves their law. The paper proves the theorem instead by induction on t with one-step conditioning. $\square$