

The Median Dynamics and 2-Choices

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Abstract

We formalize the median dynamics of Doerr, Goldberg, Minder, Sauerwald and Scheideler (SPAA 2011): every node adopts the median of its own value and the values of two random nodes. The median commutes with monotone maps, so thresholding the process at any value gives the binary median process, i.e. 2-Choices, driven by the same samples. For two values we prove consensus from a gap of order $\sqrt{n \log n}$ within $O(\log n)$ rounds with high probability, then consensus from any binary configuration and, by a union bound over the thresholds, from any configuration (Theorem 1 of the paper, without adversary). With an odd number of equally supported values, the middle one wins within $O(\log m + \log \log n)$ rounds (the odd case of Theorem 21).

1 The model

Definition 1 (Median dynamics). Values lie in a linear order. In a round every node samples two nodes uniformly at random (independently, with replacement) and adopts the median of its own value and the two sampled values. With two values this is 2-Choices: a node keeps its value unless both samples hold the other one.

2 Structure

Theorem 2 (Threshold reduction). *The median commutes with monotone maps; hence for every threshold θ the indicator of $\{x_v \geq \theta\}$ evolves, round by round and for the same samples, as the binary median process.*

Proof. $\text{med}(a, b, c) = \max(\min(a, b), \min(\max(a, b), c))$, and monotone maps commute with min and max. $\square$

Theorem 3 (Validity, range, consensus). *Every new value is a current value, values stay within any interval containing all current values, consensus configurations are fixed, and consensus is reached almost surely.*

Proof. The median is one of its arguments; the round in which every node samples node 0 twice gives consensus and has positive probability; conclude with the shared absorption theorem. $\square$

Theorem 4 (Binary expectation). *With two values and a fraction p of **true**, the expected number of **true** after one round is $n(3p^2 - 2p^3)$, the same cubic as for 3-Majority.*

Proof. A **true** node stays unless both samples are **false**, a **false** node switches iff both samples are **true**. $\square$

3 Consensus from a vanishing bias

Theorem 5 (2-Choices, w.h.p.). *If $\log n \geq 128$ and the **true** nodes outnumber the **false** ones by at least $128\sqrt{n \log n}$, all nodes hold **true** after $\lceil 128 \log n \rceil$ rounds with probability at least $1 - 128/n$.*

Proof. Phases as in the plurality package: while it is small, the gap grows by a factor $5/4$ per phase except with probability $O(n^{-2})$ (Bernstein); then the minority shrinks by $7/8$ per phase down to $O(\log n)$; a final round reaches consensus except with probability $O(n^{-1/2})$ (Markov). The phases are combined with the nested-phases lemma of the shared layer. The threshold $\log n \geq 128$, i.e. $n \geq e^{128}$, comes from these crude constants. $\square$

4 Consensus from any configuration

Theorem 6 (2-Choices from any start). *There is a constant $C > 0$ such that, whenever $\log n \geq C$, from every binary configuration (possibly perfectly balanced) all nodes agree after $\lceil C \log n \rceil$ rounds except with probability at most C/n .*

Proof. Take $C = 2^{18}$ and write g for the number of **true** nodes minus the number of **false** nodes. *Escape:* the potential $e^{-|g|/(256\sqrt{n})}$ contracts in expectation by $e^{-1/65536}$ per round, up to an additive error $e^{1/131072 - \sqrt{n}/512}$; away from balance by Hoeffding's lemma applied at every node, near balance because the next gap has variance at least $n/10$, so by Paley–Zygmund it leaves $[-\sqrt{n}/4, \sqrt{n}/4]$ with probability at least $9/64$. Iterating over $\lceil 2^{17} \log n \rceil$ rounds and applying Markov's inequality, $|g| \geq 128\sqrt{n \log n}$ except with probability $2/n$. *Consensus:* Theorem 5, applied to the configuration or to its flip, finishes within $\lceil 128 \log n \rceil$ rounds except with probability $128/n$. $\square$

Theorem 7 (Reduction to two values). *If every binary configuration fails to reach consensus within T rounds with probability at most ε , then a configuration with m distinct values fails with probability at most $(m - 1)\varepsilon$.*

Proof. If a run has not reached consensus, two nodes hold values $a < b$, and the threshold configuration $\{x_v \geq b\}$, which follows the same rounds by Theorem 2, has not reached consensus either. Take a union bound over the $m - 1$ thresholds above the minimum value. $\square$

Theorem 8 (Consensus from any configuration). *There is a constant $C > 0$ such that, whenever $\log n \geq C$, from every configuration with values in a linear order (any number of distinct values) all nodes agree after $\lceil C \log n \rceil$ rounds except with probability at most C/n . This is Theorem 1 of Doerr et al. without the adversary.*

Proof. Consensus is absorbing, so three blocks of $\lceil C_0 \log n \rceil$ rounds turn the binary failure bound C_0/n of Theorem 6 into $(C_0/n)^3$. The union bound of Theorem 7 over at most $n - 1$ thresholds gives $n(C_0/n)^3 \leq (3C_0 + 3)/n$, since $C_0^3 \leq n$ when $\log n \geq 3C_0 + 3$. $\square$

5 Many values: the middle one wins fast

Theorem 9 (Fast binary consensus from a large gap). *There is a constant $C > 0$ such that, whenever $\log n \geq C$, if the **true** nodes outnumber the **false** ones by at least $\Delta \geq C\sqrt{n \log n}$, all nodes hold **true** after $\lceil C(\log(n/\Delta) + \log \log n) \rceil$ rounds with probability at least $1 - C/n$.*

Proof. Take $C = 128$ and chain one-round moves, each failing with probability at most n^{-2} , through four segments: the gap grows by a factor $5/4$ per round until the minority is below $n/4$ ($O(\log(n/\Delta))$ rounds); six rounds bring the minority below $n/8$; it then shrinks quadratically, $m \mapsto \approx 3m^2/n$ (Bernstein), down to $512 \log n$ ($O(\log \log n)$ rounds); eight more rounds give consensus. Consensus is absorbing, so the remaining rounds are harmless, and a union bound over the rounds gives the failure probability. $\square$

Theorem 10 (Margins around a value). *There is a constant $C > 0$ such that, whenever $\log n \geq C$: if, for a value v , the nodes holding a value $\geq v$ outnumber those holding a value $< v$ by at least $\Delta \geq C\sqrt{n \log n}$, and symmetrically the nodes holding a value $\leq v$ outnumber those holding a value $> v$ by at least Δ , then all nodes hold v after $\lceil C(\log(n/\Delta) + \log \log n) \rceil$ rounds with probability at least $1 - 2C/n$.*

Proof. The indicators of $\{x_u \geq v\}$ and of $\{x_u \leq v\}$ both evolve as 2-Choices with the same samples (the second because the median is self-dual), with gaps the two margins. When both reach consensus on **true**, all nodes hold v ; conclude with Theorem 9 (with the same $C = 128$) and a union bound. $\square$

Theorem 11 (An odd number of equally supported values). *There is a constant $C > 0$ such that, whenever $\log n \geq C$: if the nodes hold $2k + 1$ distinct values, each held by $n/(2k + 1)$ nodes, and $C(2k + 1)\sqrt{n \log n} \leq n$, then the middle value is held by all nodes after $\lceil C(\log(2k + 1) + \log \log n) \rceil$ rounds with probability at least $1 - C/n$. This is the deterministic core of the odd case of Theorem 21 of Doerr et al.*

Proof. Both margins around the middle value equal $n/(2k + 1) \geq C_0\sqrt{n \log n}$, and $\log(n/\Delta) = \log(2k + 1)$; apply Theorem 10 with $C = 2C_0$. $\square$