

Simple Dynamics for Plurality Consensus

Leanamics contributors

October 7, 2026

Abstract

n anonymous nodes each support one of k colors. In every round each node samples three nodes uniformly at random (with repetition, possibly itself) and adopts the majority color among them, or the first one if all three differ. If some color m leads every other color by a sufficiently large *bias* s , the process reaches consensus on m . The paper proves that this takes $O(\lambda \log n)$ rounds w.h.p. whenever $c_m \geq n/\lambda$ and $s \geq c\sqrt{\lambda n \log n}$ (Theorem 24), that $\Omega(k \log n)$ rounds are needed from balanced configurations, that 3-majority is essentially the only 3-input rule that solves plurality consensus, and that sampling $h > 3$ nodes speeds the process up by at most a factor $O(h^2)$.

The upper bound (Section 3), including the headline bound $O(\min\{k, (n/\log n)^{1/3}\} \log n)$ of Corollary 26, is fully formalized, with no **sorry**, on the finite probability layer of the shared **dynamics** package: all randomness is uniform over finite types, multi-round probabilities are iterates of a finite Markov kernel, and the concentration inequalities (Chernoff, Hoeffding, Bernstein) are proved from scratch. The lower bounds of Section 4 are formalized as well, with two exceptions where the paper's argument does not go through: the $\Omega(k \log n)$ bound is proved for $k \leq n^{1/4-\delta}$ rather than up to $(n/\log n)^{1/4}$, and part (a) of Theorem 4.8 (Theorem 36) leaves open rules with $\Delta_r, \Delta_b \leq 1$. Observation 3.9 (an adversary) is not formalized.

1 The model

Definition 1 (Colorings and color counts). A *coloring* is a map $x: [n] \rightarrow [k]$. The number of nodes of color j is $c_j = |\{v: x(v) = j\}|$, and $\sum_j c_j = n$. The vector $c = (c_1, \dots, c_k)$ is the paper's k -color distribution (k -cd).

Definition 2 (3-input rules and the 3-majority rule). A *3-input rule* is a map $f: [k]^3 \rightarrow [k]$ with $f(a, b, c) \in \{a, b, c\}$. The *3-majority rule* is $\text{maj}_3(a, b, c) = b$ if $b = c$ and a otherwise; it returns the majority color, and the first color if all three differ.

Definition 3 (One round and the process). A round is a map $r: [n] \rightarrow [n]^3$ (the three samples of every node), drawn uniformly. The next coloring under rule f is $v \mapsto f(x(r_v^1), x(r_v^2), x(r_v^3))$. The process after T rounds folds this step over T independent uniform rounds.

Definition 4 (Consensus). x is in *consensus* on m if every node has color m , i.e. $c_m = n$. For every 3-input rule, consensus is absorbing.

Definition 5 (The Markov kernel). The process with rule f is the finite Markov kernel on colorings whose row at x is the law of one round of f from x . Its T -step event probabilities are the expectations over T independent uniform rounds.

2 The next expected coloring

Lemma 6 (Adoption probability). *The new count of color j is a sum of n independent indicators, one per node. Under 3-majority, a node adopts j with probability $x_j^2 + x_j(1 - \sum_h x_h^2)$, where $x_h = c_h/n$.*

Proof. The three samples are independent with law $(x_h)_h$. Summing the indicator of $\text{maj}_3(a, b, c) = j$ first over c , then b , then a gives the polynomial. $\square$

Lemma 7 (Lemma 2.1, next expected coloring). *For every coloring and every color j ,*

$$\mu_j(c) = \mathbb{E}[C_{j,t+1} \mid C_t = c] = c_j \left(1 + \frac{1}{n^2} \left(nc_j - \sum_h c_h^2 \right) \right).$$

Proof. Multiply the adoption probability of Lemma 6 by n . $\square$

3 The upper bound

3.1 Plurality, bias, α and γ

Definition 8 (Section 3 quantities). For a k -cd c with $\sum_h c_h = n$: $m(c) = \max_h c_h$, $M(c) = \{j : c_j = m(c)\}$, the *bias* $s(c) = m(c) - \max_{h \notin M(c)} c_h$ if $|M(c)| = 1$ and 0 otherwise, $\alpha(c) = (n - m(c))s(c)/n^2$ and $\gamma(c) = (nm(c) - \sum_h c_h^2)/n^2 - \alpha(c)$.

Lemma 9 (Every other color trails by the bias). *If $c_m = m(c)$ then $c_h + s(c) \leq m(c)$ for every $h \neq m$, with equality for some h when $k \geq 2$. Moreover $\gamma(c)n^2 = \sum_{h \neq m} c_h(m(c) - s(c) - c_h)$.*

Lemma 10 (Lemma 3.1). *For every k -cd c : (a) $0 \leq s(c) \leq m(c) - (n - m(c))/(k - 1)$; (b) $0 \leq \alpha(c) \leq \min\{s(c)/n, 1/4\}$; (c) $0 \leq \gamma(c) \leq 1/8$.*

Proof. (a) The $k - 1$ other colors each have at most $m(c) - s(c)$ nodes. (b) $(n - m)m \leq n^2/4$ and $s \leq m$. (c) Use the identity of Lemma 9: every summand is nonnegative, the color attaining the bias contributes 0, and the rest is at most $d(n - m - d) \leq n^2/8$ with $d = m(c) - s(c)$. $\square$

Lemma 11 (Lemma 3.2). *Let $m \in M(c)$ and let $\ell \neq m$ have the most nodes among the other colors. Then (a) $\mu_m(c) = c_m(1 + \gamma + \alpha)$; (b) $\mu_j(c) = c_j(1 + \gamma + \alpha - (m(c) - c_j)/n)$ for every j ; (c) $\mu_\ell(c) = c_\ell(1 + \gamma + \alpha - s(c)/n)$.*

3.2 Concentration

Lemma 12 (Concentration for sums of independent coordinates). *For $X = \sum_i Y_i(\omega_i)$ with independent uniform coordinates: Bernstein's inequality $\mathbb{P}(X \geq \mathbb{E}X + \lambda) \leq \exp(-\lambda^2/(2\sigma^2(1 + b\lambda/(3\sigma^2))))$ when $Y_i - \mathbb{E}Y_i \leq b$ and $\sum_i \text{Var } Y_i \leq \sigma^2$; Hoeffding's inequality for $\{0, 1\}$ coordinates; the multiplicative Chernoff lower tail $\mathbb{P}(X \leq (1 - \delta)\mu) \leq \exp(-\delta^2\mu/2)$; and Markov's inequality $\mathbb{P}(X \geq 1) \leq \mathbb{E}X$ for $\{0, 1\}$ coordinates.*

3.3 Growth of the bias

Lemma 13 (Lemma 3.3, increasing rate of the bias). *If $M(c) = \{m\}$ then for every $j \neq m$,*

$$\mathbb{P}\left(C_{m,t+1} - C_{j,t+1} \leq s(c) \left(1 + \gamma(c) + \frac{c_m \alpha(c)}{2s(c)} \right) \mid C_t = c\right) \leq \exp\left(-\frac{c_m \alpha(c)^2}{25}\right).$$

Proof. By Lemmas 10 and 11, $\mathbb{E}[C_{m,t+1} - C_{j,t+1}] \geq s(1 + \gamma) + c_m \alpha$. The difference is a sum of per-node variables in $\{-1, 0, 1\}$ with variance at most $3c_m$; apply Bernstein's inequality with $b = 2$ and $\lambda = c_m \alpha / 2$. $\square$

Lemma 14 (Lemma 3.4). *If $M(c) = \{m\}$ then $\mathbb{P}(C_{m,t+1} \leq c_m(1 + \gamma + \alpha/2) \mid C_t = c) \leq \exp(-c_m \alpha^2 / 11)$.*

Proof. The multiplicative Chernoff lower tail with $\delta = \alpha / (2(1 + \gamma + \alpha))$. $\square$

Lemma 15 (Lemma 3.5, large plurality and large bias). *If $M(c) = \{m\}$, $0 < \lambda \leq 2/3$, $\lambda n \leq c_m \leq (2/3)n$ and $s(c) \geq 22\sqrt{(1/\lambda)n \log n}$, then for every $j \neq m$*

$$\mathbb{P}(C_{m,t+1} - C_{j,t+1} \leq s(c)(1 + \lambda/6)) \leq 1/n^2 \quad \text{and} \quad \mathbb{P}(C_{m,t+1} \leq c_m) \leq 1/n^2.$$

Proof. Under the hypotheses $c_m \alpha / (2s) \geq \lambda/6$ and $c_m \alpha^2 / 25 \geq 2 \log n$; conclude with Lemmas 13 and 14. $\square$

3.4 Saturation

Lemma 16 (Lemma 3.6). *For every $m \in M(c)$, writing $\bar{C}_{m,t+1} = n - C_{m,t+1}$,*

$$(n - c_m) \left(1 - \frac{c_m^2}{n^2}\right) \leq \mathbb{E}[\bar{C}_{m,t+1} \mid C_t = c] \leq (n - c_m) \left(1 - \frac{s(c)c_m}{n^2}\right).$$

Lemma 17 (Lemma 3.7, very large plurality). *Let $\log n \geq 40$, $M(c) = \{m\}$. (i) If $s(c) \geq n/3$ and $n - c_m \geq n^{1/4} \log n$ then $\mathbb{P}(\bar{C}_{m,t+1} \geq \frac{17}{18}(n - c_m)) \leq 1/n^2$. (ii) If $n - c_m < n^{1/4} \log n$ then $\mathbb{P}(\bar{C}_{m,t+1} > 0) \leq n^{-1/5}$ and $\mathbb{P}(\bar{C}_{m,t+1} \geq n^{1/4} \log n) \leq 1/n^2$.*

Proof. The paper omits this proof. (i) By Lemma 16 and $s c_m \geq n^2/9$, $\mathbb{E}\bar{C}_{m,t+1} \leq \frac{8}{9}(n - c_m)$; apply the closed-form Chernoff upper tail at ratio $17/16$. (ii) Here $s(c) \geq n - 2(n - c_m)$, so $\mathbb{E}\bar{C}_{m,t+1} \leq 3(n - c_m)^2/n \leq 3 \log^2 n e^{-(\log n)/2}$; apply Markov's inequality for the first bound and the Chernoff upper tail for the second. $\square$

3.5 Nested phases and the main theorem

Lemma 18 (Lemma A.4, nested phases). *Let $A_1 \supseteq \dots \supseteq A_T$ be sets of states of a finite Markov chain such that from every state of A_i the chain stays in A_i with probability $\geq 1 - \varepsilon$ and, for $i < T$, moves into A_{i+1} with probability $\geq 1 - \nu$. Then from any state of A_1 , after ℓT steps the chain is in A_T with probability at least $1 - T(\ell\varepsilon + \nu^\ell)$.*

Proof. The paper omits this proof. Phase by phase: within ℓ steps the chain leaves A_i with probability at most $\ell\varepsilon$, and fails to jump into A_{i+1} in each of the ℓ steps with probability at most ν^ℓ . The hypothesis $\varepsilon \leq \nu$ of the paper is not needed. $\square$

Definition 19 (The phases). Fix m , λ , Λ and let $q = 1 + 1/(6\lambda)$, $B = n^{1/4} \log n$. *Growth phase i : $c_m > \frac{2}{3}n$, or $M(c) = \{m\}$, $c_m \geq n/\lambda$ and $s(c) \geq \Lambda q^i$. Saturation phase j : $n - c_m < \max\{\frac{n}{3}(\frac{17}{18})^j, B\}$. Consensus: $c_m = n$.*

Remark 20. The paper's saturation phases also require $s(c) \geq n/3$, which Lemma 17 does not show to be preserved. Here the saturation phases only bound $n - c_m < n/3$, which already implies $M(c) = \{m\}$ and $s(c) \geq n/3$ (Lemma 21).

Lemma 21 (A clear majority). *If $n - c_m < n/3$ then $M(c) = \{m\}$ and $s(c) \geq n/3$. If m leads every other color by at least $g > 0$ and $c_m \geq g$, then $M(c) = \{m\}$ and $s(c) \geq g$. A color without nodes stays without nodes.*

Lemma 22 (One round of each phase). *Let $\log n \geq 40$, $k \geq 2$, $\lambda \geq 3$ and $\Lambda \geq 22\sqrt{\lambda n \log n}$. From growth phase i the chain enters growth phase $i + 1$ with probability $\geq 1 - 1/n$. From saturation phase j it enters saturation phase $j + 1$ with probability $\geq 1 - 1/n^2$. From fewer than B dissenting nodes it reaches consensus with probability $\geq 1 - n^{-1/5}$, and consensus is kept with probability 1. The phase sets are nested, and growth phase i lies in saturation phase 0 once $\Lambda q^i > n$.*

Proof. Growth, case $c_m > \frac{2}{3}n$: saturation phase 0 moves into saturation phase 1, which lies in $\{c_m > \frac{2}{3}n\}$. Growth, otherwise: apply Lemma 15 with $1/\lambda$ to each other color j that currently has a node, and to the growth of c_m . There are fewer than $n - c_m$ such colors, so a union bound gives failure probability at most n/n^2 . Colors without nodes stay empty, and the color attaining the bias has a node (otherwise $c_m = n$), so after a successful round m leads every other color by $s(c)q$ and Lemma 21 applies. Saturation: Lemma 17(i) above B dissenters and Lemma 17(ii) below. $\square$

Lemma 23 (Number of phases). *With $T_1 = \lceil \log n / \log q \rceil$ growth phases, $T_2 = \lceil \log n / \log(18/17) \rceil + 1$ saturation phases and one consensus phase, $T = T_1 + T_2 + 1 \leq 13\lambda \log n$ for $\lambda \geq 3$ and $\log n \geq 40$.*

Proof. $\log(1 + x) \geq x/(1 + x)$ gives $\log q \geq 1/(6\lambda + 1)$ and $\log(18/17) \geq 1/18$. $\square$

Theorem 24 (Theorem 3.8, the general upper bound). *Let $\log n \geq 40$, $k \geq 2$ and $\lambda \geq 3$. If $M(c) = \{m\}$, $c_m \geq n/\lambda$ and $s(c) \geq 22\sqrt{\lambda n \log n}$, then after $10T \leq 130\lambda \log n$ rounds all nodes support m with probability at least $1 - 11T/n \geq 1 - 143\lambda \log n/n$.*

Proof. Apply Lemma 18 to the kernel of 3-majority with the T_1 growth phases, the T_2 saturation phases and consensus, $\varepsilon = 1/n$, $\nu = n^{-1/5}$ and $\ell = 10$, so that $\nu^\ell = n^{-2}$. The initial coloring is in growth phase 0; growth phase T_1 lies in saturation phase 0 because $\Lambda q^{T_1} \geq 22n$; saturation phase $T_2 - 1$ lies below B dissenters because $\frac{n}{3}(\frac{17}{18})^{T_2-1} \leq \frac{1}{3}$. $\square$

Remark 25. The paper states the failure probability as $11T/n^2$; the extra factor n comes from the union bound over competing colors in Lemma 22, which the paper omits. The paper's hypothesis $\lambda < \sqrt{n}$ is not used; it only makes the hypothesis on $s(c)$ satisfiable.

Corollary 26 (Corollary 3.10). *If $M(c) = \{m\}$ and $s(c) \geq 22\sqrt{\min\{2k, (n/\log n)^{1/3}\} n \log n}$, then 3-majority reaches consensus on m in $O(\min\{2k, (n/\log n)^{1/3}\} \log n)$ rounds w.h.p. Precisely, for $\log n \geq 40$ and $k \geq 2$: at most $130\lambda \log n$ rounds with probability at least $1 - 143\lambda \log n/n$.*

Proof. Take $\lambda = \min\{2k, (n/\log n)^{1/3}\}$. If $\lambda = 2k$ then $c_m \geq n/k$ by pigeonhole; otherwise $\lambda^3 = n/\log n$, so $22\sqrt{\lambda n \log n} = 22n/\lambda$ and $c_m \geq s(c) \geq n/\lambda$. Moreover $\lambda \geq 3$ because $k \geq 2$ and $n = e^{\log n} \geq 27 \log n$. $\square$

Corollary 27 (Corollaries 3.11 and 3.12). *If $c_m \geq n/\log^\ell n$ and $s(c) \geq 22\sqrt{n \log^{\ell+1} n}$, consensus takes $O(\log^{\ell+1} n)$ rounds w.h.p. If $c_m \geq n/\beta$ and $s(c) \geq 22\sqrt{\beta n \log n}$ for a constant $\beta \geq 3$, it takes $O(\log n)$ rounds w.h.p.*

Proof. Take $\lambda = \log^\ell n$ (with $\ell \geq 1$, so that $\lambda \geq 3$) and $\lambda = \beta$ in Theorem 24. $\square$

3.6 Two colors

Theorem 28 (Binary case). *With $k = 2$ the set of nodes of color 1 evolves exactly as the binary 3-majority process of the `three_majority` package. Hence, if $c_1 \geq \frac{3}{5}n$ and $\log n \geq 30$, all nodes support color 1 after $O(\log n)$ rounds with probability at least $1 - 500/n$.*

Corollary 29 (Two opinions from a vanishing bias). *In the binary 3-majority process, if the initial majority leads by at least $22\sqrt{3n \log n}$ nodes (a fraction $\frac{1}{2} + O(\sqrt{\log n/n})$) and $\log n \geq 40$, all nodes adopt it within at most $390 \log n$ rounds with probability at least $1 - 429 \log n/n$.*

Proof. Theorem 24 with $k = 2$ and $\lambda = 3$: the majority color is the unique plurality, holds more than $n/2 \geq n/3$ nodes, and its bias is the gap; the conclusion transfers to the binary process through Theorem 28. $\square$

4 Lower bounds

4.1 Lower bound for 3-majority

Lemma 30 (Escaping a moving target). *If a round-based process starts in G_0 and, for every $t < T$, one round from any state of G_t misses G_{t+1} with probability at most p , then after T rounds it lies outside G_T with probability at most Tp .*

Proof. Induction on T , conditioning on the first round. $\square$

Lemma 31 (Lemma 4.1). *Let $k\sqrt{n \log n} \leq b \leq n/k$. If $c_j \leq n/k + b$, then $\mathbb{P}(C_{j,t+1} \geq n/k + (1 + 3/k)b \mid C_t = c) \leq 1/n^2$.*

Proof. By Cauchy-Schwarz $\sum_h c_h^2 \geq n^2/k$, so $\mu_j \leq c_j(1 + c_j/n - 1/k)$. Writing $c_j = n/k + a$ with $a \leq b$, this is at most $n/k + (1 + 1/k)b + b^2/n \leq n/k + (1 + 2/k)b$. Hoeffding's inequality with deviation $b/k \geq \sqrt{n \log n}$ gives $1/n^2$. $\square$

Theorem 32 (Theorem 4.2, lower bound for 3-majority). *Let $k \geq 3$ and suppose every color starts with at most $n/k + b$ nodes, where $b \geq k\sqrt{n \log n}$. If $b(1 + 3/k)^T \leq n/k$, in particular if $T \leq \frac{k}{3} \log \frac{n}{kb}$, then the coloring after T rounds is monochromatic with probability at most kT/n^2 .*

Proof. For each color apply Lemma 30 to the thresholds $n/k + b(1 + 3/k)^t$, with Lemma 31 for each round. A monochromatic coloring has a color with $n > 2n/k$ nodes, above its final threshold; sum over the k colors. $\square$

Remark 33. With $b = \max\{(n/k)^{1-\varepsilon}, k\sqrt{n \log n}\}$ this gives $\Omega(k \log(n/(kb)))$ rounds, which is $\Omega(k \log n)$ for $k \leq n^{1/4-\delta}$. The paper states $\Omega(k \log n)$ for all $k \leq (n/\log n)^{1/4}$, but at that end $n/k = k\sqrt{n \log n}$ and the argument gives nothing. The paper's proof also chooses the thresholds after seeing the trajectory; the union bound needs the fixed schedule used here.

4.2 Which 3-input rules solve plurality consensus

Definition 34 (Rules, Δ , δ and solvers). A rule is *conservative* if it returns one of its inputs. It has the *clear-majority* property if it returns the majority color whenever two inputs agree. For colors $r \neq b$, Δ_r counts the orderings of (r, r, b) on which the rule returns r . For distinct r, g, b , δ_c counts the orderings of (r, g, b) on which it returns c ; the rule is *uniform* if $\delta_r = 2$ for all distinct triples. It is an (s, ε) -solver if from every coloring whose unique plurality color m has bias at least s , consensus on m is eventually reached with probability at least $1 - \varepsilon$.

Lemma 35 (Consensus needs survival). *Under a conservative rule a color without nodes never returns. Hence for all T, T_0 , the probability of consensus on m at time T is at most the probability that m has a node at time T_0 .*

Theorem 36 (Theorem 4.8(a), clear majority). *Let $8 \mid n$. With only colors r, b present and x the fraction of r , a node adopts r with probability $p(x)$, where $p(x) - x = x(1-x)(x(\Delta_r - 2) + (1-x)(2 - \Delta_b))$. If $\Delta_r \leq 2 \leq \Delta_b$, the number of r nodes is a supermartingale, so from $5n/8$ nodes of r (bias $n/4$) consensus on r has probability at most $5/8$ at every time: the rule is not an $(n/4, 1/4)$ -solver. Consequently, for an $(n/4, 1/4)$ -solver and every pair $r \neq b$, either $\Delta_r = \Delta_b = 3$ or $\Delta_r, \Delta_b \leq 1$.*

Proof. $\mathbb{E}[X_T] \leq X_0$ by induction on T , and $\mathbb{P}(X_T = n) \leq \mathbb{E}[X_T]/n$. $\square$

Remark 37. The paper claims that an $(n/4, 1/4)$ -solver has the clear-majority property, i.e. that $\Delta_r, \Delta_b \leq 1$ is also impossible. Its proof uses the supermartingale property, but checks it only when red is the majority; for $\Delta_r, \Delta_b \leq 1$ it fails when red is the minority, because the process is pulled to the interior point $x^* = (2 - \Delta_b)/(4 - \Delta_r - \Delta_b)$. That case is not formalized.

Lemma 38 (Adopting r among three colors). *Let the rule be conservative with the clear-majority property, let only r, g, b be present and $\delta_r \leq 1$. Then $p(r) \leq 3x^2 - 2x^3 + \delta_r x_r x_g x_b \leq x(3x - 2x^2 + (1-x)^2/4)$, which is at most $0.97x$ for $x \leq 2/5$; and from $x \leq 2/5$ one round ends above $2n/5$ with probability at most $e^{-9n/31250}$.*

Lemma 39 (r dies out). *Under the hypotheses of Lemma 38, starting with at most $2n/5$ nodes of r , the probability that r still has a node after T rounds is at most $0.97^T c_r + T e^{-9n/31250}$.*

Proof. Consider the chain frozen once r exceeds $2n/5$. On it, $\Phi = C_r \mathbb{1}[C_r \leq 2n/5]$ contracts by 0.97 in every state, it is frozen within T rounds with probability at most $T e^{-9n/31250}$, and it agrees with the real chain until it is frozen. $\square$

Theorem 40 (Theorem 4.8(b), uniform property). *Let $60 \mid n$ and $\log n \geq 20$. A rule with the clear-majority property that is an $(n/20, 1/4)$ -solver has the uniform property.*

Proof. If a triple is not uniform, some color r of it has $\delta_r \leq 1$, since the three counts sum to 6. Start from $23n/60$ nodes of r , $n/3$ of g and $17n/60$ of b : r is the unique plurality color with bias $n/20$. By Lemmas 39 and 35, with $T_0 = \lceil \log(8n)/\log(100/97) \rceil$, consensus on r has probability at most $1/8 + 1/8$ at every time. $\square$

Remark 41. The paper's proof (Lemma 4.10) is a sketch that treats $\delta_r = 1$ with $(\delta_g, \delta_b) \in \{(3, 2), (4, 1)\}$ and asserts that r “will not converge” by iterating a one-round estimate. The proof here covers every non-uniform triple and makes the iteration rigorous.

4.3 A lower bound for h -plurality

Definition 42 (h -plurality). Every node samples h nodes and a uniformly random ordering σ of the h sample positions, and adopts the color of the position of smallest σ -rank among those whose color is most frequent. All tied colors appear equally often, so each is chosen with the same probability.

Lemma 43 (Adoption under h -plurality). *A node adopts j with probability at most $x_j + \frac{h^2}{2} x_j^2$.*

Proof. If j wins and is seen exactly once, all h sampled colors are distinct and j is at the position ranked first, which has probability x_j . Otherwise j is seen at least twice, and $\mathbb{P}(N_j \geq 2) \leq \mathbb{E}[N_j(N_j - 1)]/2 = \binom{h}{2} x_j^2$. $\square$

Lemma 44 (Lemma 4.11). *If $c_j \leq a \leq 2n/k$, then $\mathbb{P}(C_{j,t+1} \geq a(1+2h^2/k)) \leq \exp(-2(ah^2/k)^2/n)$, which is at most e^{-2nh^4/k^4} when $a \geq n/k$.*

Remark 45. The paper states the factor $1 + h^2/k$ but its proof gives $1 + 2h^2/k$, which is also what Theorem 46 uses. Allowing $c_j \leq a$ instead of $c_j = a$ covers colors below n/k , which the paper's proof of Theorem 46 needs but the lemma excludes.

Theorem 46 (Theorem 4.12, lower bound for h -plurality). *Let $k \geq 3$. If every color starts with at most $\frac{3}{2} \cdot \frac{n}{k}$ nodes and $T \leq \frac{k}{2h^2} \log \frac{4}{3}$, then after T rounds of h -plurality the coloring is monochromatic with probability at most kTe^{-2nh^4/k^4} .*

Proof. As for Theorem 32, with thresholds $\frac{3}{2} \cdot \frac{n}{k}(1 + 2h^2/k)^t \leq 2n/k$. □