

Weighted Synchronous Voter Dynamics

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Abstract

We formalize Sections 2.1–2.3 of Hassin and Peleg, *Distributed Probabilistic Polling and Applications to Proportionate Agreement*, Information and Computation 171 (2001), 248–268. The synchronous process uses independent neighbor samples with directional weights. On a finite connected nonbipartite undirected graph it reaches consensus with probability one, and each color wins with its initial stationary vertex weight. No path-space measure is needed: eventual probabilities are suprema of increasing finite-time consensus probabilities.

1 Shared foundation (generic library results)

Definition 1 (Finite weighted expectations). A distribution has nonnegative real weights summing to one. A kernel is a family of such distributions, one for each state. Expectations are finite sums. These are shared library definitions, not numbered results of the paper.

Theorem 2 (Stationary distribution exists). *Every finite nonempty stochastic matrix has a stationary distribution.*

Proof. The shared library uses Cesàro averages of evolved distributions and compactness of the finite probability simplex. This supplies the existence fact recalled before Lemma 2.2; uniqueness is not required. $\square$

Theorem 3 (Finite-chain absorption). *An absorbing target accessible from every state of a finite chain admits a uniform positive success probability in a fixed number of rounds. The survival probability decays geometrically in blocks and tends to zero.*

Proof. Take a common block length exceeding all finitely many access times, then a maximum survival probability strictly below one. Iterate the resulting contraction. $\square$

Theorem 4 (Finite-horizon optional stopping). *If an observable equals φ_A on A , φ_B on B , lies in $[\varphi_B + m, \varphi_B + M]$ elsewhere, and keeps its expectation up to time T , then $\varphi(x_0) - \varphi_B - (\varphi_A - \varphi_B)\mathbb{P}(X_T \in A)$ lies between m and M times the probability of lying in neither event at time T .*

Proof. A shared library result (roadmap FND-4): decompose the observable pointwise into its values on A and off $A \cup B$, take expectations, and bound the second part. $\square$

2 Section 2.1: model

Definition 5 (Synchronous copying). A configuration is $s : V \rightarrow C$. Independently sample $r(i)$ according to row H_i , then set the next configuration to $i \mapsto s(r(i))$.

Lemma 6 (Configuration transition formula). *For configurations s, t , the transition probability is*

$$P_{st} = \prod_i \sum_{j:s(j)=t(i)} H_{ij}.$$

For Boolean colors a white target vertex contributes $\sum_j H_{ij}s(j)$; a black target contributes its complement, giving the paper's product formula.

Proof. The configuration distribution is the pushforward of the independently sampled neighbors. Factor its equality indicator into a product over vertices. $\square$

3 Section 2.2: absorption and consensus probability

Lemma 7 (Monochromatic-edge propagation). *On a connected nonbipartite graph every Boolean configuration has a possible finite path to a constant configuration.*

Proof. A coloring without a monochromatic edge would be a proper two-coloring. Starting with a monochromatic edge, each vertex of its region can keep that color by sampling an internal neighbor. Connectedness supplies an edge leaving every proper region. One more vertex can copy across that edge. Induction on the number of vertices outside the region gives consensus. $\square$

Lemma 8 (Lemma 2.1: absorption). *Assume $H_{ij} > 0$ whenever i, j are adjacent; extra support, such as self-loops, is allowed. Both constant configurations are absorbing, and the probability of remaining nonconstant tends to zero. This is the finite-time survival formulation of the paper's transience claim.*

Proof. Every possible copying round has positive probability. Thus graph propagation gives accessibility. Apply the shared finite-chain absorption theorem. $\square$

Lemma 9 (Lemma 2.3: invariant stationary weight). *For any stationary distribution π , the expected white mass at every finite time is the initial mass $\sum_i \pi_i s(i)$. No graph connectivity assumption is imposed on this invariant.*

Proof. A vertex's expected next color indicator is $\sum_j H_{ij}s(j)$. Interchanging finite sums and using $\pi H = \pi$ proves one-step preservation; induction proves the iterated statement. $\square$

Lemma 10 (Lemma 2.2: limiting identification). *The discrepancy between expected stationary white mass and all-white probability lies between zero and nonconsensus probability. Consequently all-white probability converges to initial stationary white mass. The implementation combines the paper's limiting identification with the invariant from Lemma 2.3 when stating the limit.*

Proof. Pointwise, the white mass agrees with the all-white indicator on constant configurations and lies in $[0, 1]$ elsewhere. Finite-horizon optional stopping, with the two consensus configurations as targets and the invariant of Lemma 2.3, bounds the discrepancy; squeeze it using Lemma 2.1. $\square$

Theorem 11 (Theorem 2.1). *Define eventual all-white probability as the supremum of the increasing finite-time all-white probabilities. It equals the initial stationary white mass.*

Proof. Consensus is absorbing, so the probabilities increase and are bounded by one. Their supremum is their limit. Identify this limit using Lemma 2.2. $\square$

Theorem 12 (Corollary 2.2 and regular graphs). *For uniform neighbor sampling, $\pi_i = d_i/(2|E|)$, so all-white consensus has probability $\sum_{i:s(i)=1} d_i/(2|E|)$. On a regular graph this is the initial fraction of white vertices.*

Proof. The degree-sum formula normalizes these weights. In the stationarity equation, the degree cancels the reciprocal sampling degree; undirected adjacency gives the remaining column sum. Equal degrees then give uniform vertex weights. $\square$

4 Section 2.3: multiple colors

Theorem 13 (Consensus in a specified color). *For a finite color set and any specified color c , its eventual consensus probability is $\sum_{i:s(i)=c} \pi_i$. These probabilities sum to one.*

Proof. The color-indicator projection commutes with every copying step and with iterated expectations. Its all-white event is exactly all- c consensus. Apply Theorem 2.1 to this Boolean projection. $\square$

5 Section 2.4 on the complete graph: coalescing walks and consensus time

Definition 14 (Rounds, backward map, Wright–Fisher sampling). A run applies rounds $r_1, \dots, r_T: V \rightarrow V$ in order. Its backward map is $r_1 \circ r_2 \circ \dots \circ r_T$. The disagreement indicator is 0 on constant configurations and 1 otherwise. Wright–Fisher sampling picks every row uniformly on V , self included, i.e. the complete graph with self-loops.

Lemma 15 (Duality with coalescing walks). *After rounds $r_1, \dots, r_T$ the configuration is $s \circ r_1 \circ \dots \circ r_T$: running the voter forward reads the initial colours through the backward map, whose point images are coalescing random walks. If the backward map is constant, the configuration is a consensus.*

Proof. Induction on the list of rounds, since one round maps s to $s \circ r$. $\square$

Lemma 16 (Rounds are independent and uniform). *Under Wright–Fisher sampling a round is uniform on $V \rightarrow V$, so the T -step configuration kernel is the average over T independent uniform rounds.*

Proof. The product weights are all n^{-n} ; induct on T , peeling off the first round. $\square$

Lemma 17 (Meeting probability). *Two coordinates of a uniform map $V \rightarrow V$ agree with probability $1/n$. Hence the backward walks from two distinct vertices have not met after T rounds with probability exactly $(1 - 1/n)^T$.*

Proof. Peel off the first round, swap it with the remaining average, and multiply by the one-round probability of not meeting; induct on T . $\square$

Theorem 18 (Consensus time). *The probability that no consensus is reached after T rounds is at most $(n - 1)(1 - 1/n)^T$. Consequently consensus is reached within $2n \log n$ rounds with probability at least $1 - 1/n$.*

Proof. Consensus holds once every backward walk has met the walk of a fixed vertex; apply the union bound over the other $n - 1$ vertices and the meeting probability, then $(1 - 1/n)^T \leq e^{-T/n} \leq n^{-2}$ for $T \geq 2n \log n$. $\square$

6 Self-loops, the lazy voter and Wright–Fisher (VOT-2)

Lemma 19 (Propagation through the kernel support). *Call a round possible for H when every vertex samples a vertex of positive weight in its row. On a connected graph whose edges are charged by H , a monochromatic region in which every vertex charges a vertex of the region grows to consensus through possible rounds. If $H_{vv} > 0$ for one vertex v , the region $\{v\}$ qualifies, so nonconsensus probability tends to zero, bipartite graphs included.*

Proof. As in Lemma 2.1: a boundary vertex copies its neighbour in the region, vertices of the region copy inside it, and all other vertices copy any vertex of positive weight. Possible rounds have positive probability, and the finite-chain absorption theorem applies. $\square$

Theorem 20 (Remark, p. 254). *On a connected graph, bipartite or not, if H charges every edge and $H_{vv} > 0$ for some vertex v , then consensus in colour c has probability $\sum_{i:s(i)=c} \pi_i$ for every stationary π .*

Proof. Lemma 2.2 only needs nonconsensus probability to vanish; then project colours as in Section 2.3. $\square$

Theorem 21 (Lazy voter). *The lazy kernel $(I + H)/2$ has the stationary distributions of H and positive self-loops. On any connected graph the lazy voter $(I + D^{-1}A)/2$ reaches consensus in colour c with probability $\sum_{i:s(i)=c} d_i/2m$.*

Proof. Apply the Remark to $(I + H)/2$; for uniform neighbour sampling use the degree distribution. $\square$

Theorem 22 (Neutral Wright–Fisher fixation). *If every offspring picks a uniform parent, itself included, allele a fixes with probability c_a/n , where c_a is its initial number of copies.*

Proof. The uniform distribution is stationary and every entry is positive. For $n \geq 3$ the complete graph is nonbipartite and Section 2.3 applies; for every n the Remark applies. $\square$

7 Section 2.4 on connected graphs: meeting time and $O(n^3 \log n)$ consensus

Definition 23 (Two-token walk, lazy sampling). For a sampling kernel H , two tokens move with the rounds of the voter dynamics: the token at x moves to $r(x)$, where r is drawn from the independent product of the rows of H . Off the diagonal the tokens make independent H -steps; on the diagonal they move together. The lazy kernel of a graph samples the vertex itself with probability $1/2$ and otherwise a uniformly random neighbour. Rounds can also be composed on the left, $W \mapsto r \circ W$.

Lemma 24 (Duality for every kernel). *For every sampling kernel, the probability that no consensus is reached after T rounds is at most the sum over $v \neq u_0$ of the probability that the tokens started at v and u_0 are apart after T steps.*

Proof. The voter dynamics on maps started from the identity composes the rounds on the right; composing them on the left gives the same finite-time expectations, since the rounds are independent and identically distributed (induction on T , exchanging two finite expectations). The left composition read at two vertices is the two-token walk. Conclude with the pointwise union bound of Theorem 18. $\square$

Theorem 25 (Hassin–Peleg Theorem 2.4, tail form). *If from every pair of vertices the tokens are apart after T_0 steps with probability at most $1/2$, then consensus fails after kT_0 rounds with probability at most $(n-1)2^{-k}$.*

Proof. The diagonal is absorbing, so the probabilities of being apart are submultiplicative in blocks of T_0 steps; apply Lemma 24. $\square$

Lemma 26 (Hitting times of the lazy walk; Fact 2.3). *On a connected graph with Laplacian L and $\text{vol} = \sum_z d_z$, the system $h(y) = 0$, $(Lh)(x) = 2d_x$ for $x \neq y$ has a nonnegative solution h_y , the expected hitting time of y for the lazy walk. It satisfies $\sum_z d_z h_y(z) - \text{vol} h_y(x) = \sum_z d_z h_x(z) - \text{vol} h_x(y)$ and $h_y(x) + h_x(y) \leq 4 \text{vol} (n-1) \leq 4n^3$.*

Proof. The maximum principle makes the linear system injective, hence surjective. The identity is the symmetry of L . For the bound, $\psi = h_y - h_x$ has energy $\psi^\top L \psi = 2 \text{vol} (h_y(x) + h_x(y))$; apply Cauchy–Schwarz along a path from x to y . $\square$

Lemma 27 (Sequential and synchronous walks). *Let the two tokens move one at a time, meeting being checked after each move. If a nonnegative potential F drops by one in expectation under a step of either token, the sequential walks are apart after T steps with probability at most F/T . For a lazy kernel, the synchronous tokens are apart after $T+1$ steps with probability at most the average of the indicator of starting apart and the sequential survival; hence at most $3/4$ once $T \geq 2 \sup F$.*

Proof. Two inductions on T . When the first token lands on the old position of the second, the second stays put with probability at least $1/2$. $\square$

Theorem 28 (Meeting time of lazy walks; Lemma 2.4). *On a connected graph with n vertices, two lazy walks are apart after $3(16n^3 + 1) \leq 51n^3$ steps with probability at most $1/2$, and after $51kn^3$ steps with probability at most 2^{-k} , from every pair of start vertices.*

Proof. Apply Lemma 27 to the potential $h_y(x) - \sum_z d_z h_y(z)/\text{vol} + 4n^3$ of Lemma 26, bounded by $8n^3$, then submultiplicativity. $\square$

Theorem 29 (Hassin–Peleg Theorem 2.5; Survey Theorem 8). *On every connected graph with n vertices and from every initial colouring, the lazy voter dynamics reaches consensus within $255n^3 \log n$ rounds with probability at least $1 - 1/n$.*

Proof. With $k = \lfloor T/(51n^3) \rfloor$, Lemma 24 and Theorem 28 bound the failure probability by $(n-1)2^{-k} \leq 1/n$. $\square$

8 Consensus time via conductance (VOT-5)

Source: Berenbrink, Giakkoupis, Kermarrec, Mallmann-Trenn, *Bounds on the voter model in dynamic networks*, ICALP 2016, arXiv:1603.01895 (BGKM16). The process is the lazy voter $(I + D^{-1}A)/2$ of Theorem 21.

Definition 30 (Volume, conductance, minority side, potential). $\text{vol}(S) = \sum_{u \in S} d_u$; the conductance $\varphi(G)$ is the minimum of $|\text{cut}(U, V \setminus U)|/\text{vol}(U)$ over $0 < \text{vol}(U) \leq m$; λ_u is the number of neighbours of u with another opinion; for two opinions s_t is the side of smaller volume and $\Psi = \sqrt{\text{vol}(s_t)}$.

Lemma 31 (Independent coordinates). *Resampling one coordinate of an independent point does not change its law. Consequently centred coordinate functions have additive second and third moments, and if replacing each coordinate function improves $\mathbb{E}f(z + \cdot)$ for all admissible z , then the replacement of all of them improves $\mathbb{E}f(c + \sum_i g_i(x_i))$ (BGKM16 Lemma A.1).*

Proof. Reindex the pairs (x, a) by $(x[i \mapsto a], x_i)$; induct over the set of replaced coordinates. $\square$

Lemma 32 (BGKM16 Lemma 2.1, corrected). *If $s_t \neq \emptyset$, then $\mathbb{E}[\Psi_{t+1} \mid s_t] \leq \Psi - \sum_{u \in s_t} \lambda_u d_u / (32\Psi^3) \leq \Psi - d_{\min}\varphi / (32\Psi)$. The paper prints the sum over all of V , which fails on the star $K_{1,15}$; its proof gives the sum over s_t .*

Proof. Ψ_{t+1} is at most the root of the new volume $P + \sum_u X_u$ of the opinion of s_t . Replace X_u for $u \notin s_t$ by $Y_u = \lambda_u \mathbf{1}[\text{move}]$ (concavity of $\sqrt{\cdot}$), expand $\sqrt{P + x}$ to third order, and use $\mathbb{E}\Delta' = 0$, $\mathbb{E}\Delta'^2 \geq \sum_{s_t} \lambda_u d_u / 4$, $\mathbb{E}\Delta'^3 \leq 0$. The conductance gives $\sum_{s_t} \lambda_u d_u \geq d_{\min}\varphi \text{vol}(s_t)$. $\square$

Theorem 33 (BGKM16 Lemma 2.2 and Theorem 1.1 (i), two opinions). *If $128 \text{vol}(s_0) \leq d_{\min}\varphi T$, the opinions still disagree at time T with probability at most $1/2$; in particular after $128m/(d_{\min}\varphi)$ rounds, and the expected consensus time is at most twice that. On dynamic graphs with fixed degrees the same holds once $128 \text{vol}(s_0) \leq d_{\min} \sum_{t < T} \varphi_t$.*

Proof. The drift lemma of the shared library (FND-5) with drift $d_{\min}\varphi_t/32$; restarting bounds the expected time. $\square$

Theorem 34 (BGKM16 Theorem 1.1 (i), any number of opinions). *There is a constant b such that the expected consensus time is at most $b m / (d_{\min}\varphi)$ and consensus holds within $b m / (d_{\min}\varphi)$ rounds with probability at least $1/2$, for every number of opinions.*

Proof. Project each opinion to the two-opinion process “ i against the rest”. From $\ell \leq \theta$ opinions, after $384 \text{vol}(V) / (\theta d_{\min}\varphi)$ rounds at most $5\theta/6$ remain with probability at least $1/3$ (BGKM16 Lemma 2.3). Summing over the levels $\theta_j = n(5/6)^j$ gives a geometric series; Markov’s inequality gives the probability form. $\square$

9 Examples and scope

Lemma 35 (Triangle and a nonuniform matrix). *One white vertex on a triangle wins with probability $1/3$. The stochastic matrix whose two rows are $(1/3, 2/3)$ has stationary weights $(1/3, 2/3)$, providing a nonuniform invariant example.*

Lemma 36 (Wright–Fisher sampling). *On three individuals, let every offspring pick a uniform parent, itself included. This kernel has self-loops, so it is supported beyond the edges of the triangle. Theorem 2.1 still applies, and one allele copy fixes with probability $1/3$.*

Lemma 37 (Bipartite two-cycle). *On the two-vertex graph, every vertex must copy the other vertex. The alternating coloring flips each round and returns after two rounds; neither coloring is constant. Thus connectedness alone is insufficient.*

Hassin–Peleg Theorem 2.5 is formalized for the lazy walk; for the plain walk on nonbipartite graphs it holds conditionally on a meeting-time bound (Theorem 25). Dynamic networks and extremal coalition results are future work beyond this milestone. Chernoff bounds remain in the 3-majority package until another theorem requires their extraction.